SOME RECENT RESULTS IN THE PROBLEM OF THREE BODIES*

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1. Introduction. The classical mechanics is not so much in vogue nowadays as it was some years ago. The study of quantum mechanics and the theory of relativity have claimed the major part of the attention of mathematical physicists and astronomers, but the Newtonian mechanics has by no means been completely neglected. The famous problem of three bodies has received a share of attention, and some interesting results have been obtained in the last ten years. I shall endeavor to recount some of these results in so far as they relate to the problem of three finite masses.

2. A brief survey of earlier results. To give a background to the questions taken up later, let us first summarize briefly certain earlier work on the three body problem.

The problem of three bodies can be stated very simply. Given three rigid spherical bodies, having masses, m_1, m_2, and m_3, homogeneous in concentric layers, subject to no forces except their own attractions under the Newtonian law of gravitation. Let them be in arbitrary positions and moving with arbitrary velocities at any time, t_0. We wish to find the orbit each will trace and the position of each in its orbit at any future time.

The first serious attack on the three body problem dates back to Newton. He gave general theorems on the motion of the center of mass, and the first discussion of perturbations, in Book I, Section XI of the *Principia*, which was published in 1687, just two hundred and fifty years ago. From Newton to the present year the list of those who have contributed something toward the solution of the three body problem contains the names of many of the world's most famous mathematicians. Among them are: Euler, Lagrange, Laplace, Jacobi, Gauss, Poincaré, Hill, F. R. Moulton, Levi-Civita, and Sundman.

The first particular solutions of the problem of three bodies were found by Lagrange in 1772 [1]. These are the celebrated straight line and equilateral triangle solutions. In the straight line solutions the bodies always remain on a straight line and revolve with uniform angular velocity in circles or ellipses about their common center of mass. In the equilateral triangle case they remain at the vertices of an equilateral triangle which rotates uniformly about the center of mass. The size of the triangle may change as it rotates so that each body may describe an ellipse with a focus at the center of mass of the three bodies.

3. Rotating axes. The fact that in each of these particular Lagrangian solutions the bodies may move in circles with constant angular velocity probably suggested to later writers that the positions of members of the system might

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well be given by reference to a set of axes rotating with uniform angular velocity. If the rectangular coördinates $\bar{\xi}_i$, $\bar{\eta}_i$, $\bar{\zeta}_i$ be transformed by the equations

$$\begin{aligned} \bar{\xi}_i &= \xi_i \cos \omega t - \eta_i \sin \omega t, \\ \bar{\eta}_i &= \xi_i \sin \omega t + \eta_i \cos \omega t, \\ \bar{\zeta}_i &= \zeta_i, \end{aligned}$$

ω being the angular velocity, we have the following eighteenth order system of differential equations:

$$\begin{aligned} \frac{d^2\xi_i}{dt^2} - 2\omega \frac{d\eta_i}{dt} &= X_i(\xi_i, \eta_i, \zeta_i), \\ \frac{d^2\eta_i}{dt^2} + 2\omega \frac{d\xi_i}{dt} &= Y_i(\xi_i, \eta_i, \zeta_i), \\ \frac{d^2\zeta_i}{dt^2} &= Z_i(\xi_i, \eta_i, \zeta_i), \qquad (i = 1, 2, 3), \end{aligned} \tag{1}$$

where the center of gravity is at the origin and where X_i, Y_i, Z_i are functions of ξ_i, η_i, ζ_i.

4. Solutions and integrals. Before we proceed further we should make clear the meanings of the terms "solution" and "integral." Suppose that at any instant of time, t_0, it is known that $\xi_i = a_i$, $\eta_i = b_i$, $\zeta_i = c_i$, $d\xi_i/dt = d_i$, $d\eta_i/dt = e_i$, $d\zeta_i/dt = f_i$, $i = 1, 2, 3$. Then by a solution of equations (1) we shall mean a set of equations expressing ξ_i, η_i, ζ_i and their derivatives as functions of t, satisfying equations (1) identically in t and reducing to $\xi_i = a_i, \cdots, d\zeta_i/dt = f_i$, $(i = 1, 2, 3)$, at $t = t_0$.

The function, $F(\xi_i, \eta_i, \cdots, d\zeta_i/dt, t)$, is an integral of (1) if it reduces to a constant for every value of t when any solution of (1) is substituted for the variables.

5. Integrals of equations (1). Equations (1) have 10 known integrals. These and the theorems to which they lead were known to Euler. At the present time it seems highly probable that there are no more integrals of a simple type. Bruns [**2**] and Poincaré [**3**] have made important contributions in this connection. Six of these integrals are linear functions of ξ_i, η_i, and ζ_i, and their derivatives. They can be used to eliminate ξ_2, η_2, and ζ_2 and thus reduce the order of the system to the 12th. We shall suppose that this has been done.

6. Particular solutions. Equations (1) are satisfied by $\xi_i = a_i$, $\eta_i = \zeta_i = 0$, which requires that the three bodies remain on the ξ_i-axis. They can also be satisfied by $\xi_i = a_i$, $\eta_i = b_i$, $\zeta_i = 0$, where (a_i, b_i) are the coördinates of the vertices of an equilateral triangle in the $\xi_i\eta_i$-plane.

This is an easier way of getting the straight line and triangle solutions than the method of Lagrange, but it has the disadvantage of not giving the elliptical orbits.

Obviously one important question which we may ask is whether or not these Lagrangian positions are stable; that is, if three bodies, large or small, should ever be found approximately in the proper positions on a straight line with proper velocities would the system oscillate near these positions for a considerable time or would it rapidly break up. Liouville, in 1845, answered this question for the case in which the three bodies are the sun, the earth, and the moon, assumed placed in the straight line positions; he showed that the position was unstable.

7. Stability of the straight line solutions. The method of determining the stability is as follows: In equations (1) let $\xi_i=a_i+x_i$, $\eta_i=y_i$, $\zeta_i=z_i$, then expand the right members as power series in x_i, y_i, and z_i. Drop all terms except the linear ones from the right members. The resulting system of linear, homogeneous differential equations are called the equations of variation. The z_i equations happen to be independent of the x_i and y_i equations in all the cases we shall consider except one which will be mentioned at the proper time. We have, then, for $(i=1, 2, 3)$,

$$\left\{\begin{aligned} \frac{d^2x_i}{dt^2} - 2\omega\frac{dy_i}{dt} &= A_{i1}x_1 + A_{i3}x_3 + B_{i1}y_1 + B_{i3}y_3, \\ \frac{d^2y_i}{dt^2} + 2\omega\frac{dx_i}{dt} &= C_{i1}x_1 + C_{i3}x_3 + D_{i1}y_1 + D_{i3}y_3, \end{aligned}\right. \tag{2}$$

$$\frac{d^2z_i}{dt^2} = E_{i1}z_1 + E_{i3}z_3. \tag{3}$$

These equations may be solved by setting $x_i=K_ie^{\lambda t}$, $y_i=L_ie^{\lambda t}$, $z_i=M_ie^{\lambda t}$. In order that the resulting equations (2) may have a solution different from $K_i=L_i=0$, it is necessary that their determinant vanish. This gives the following equations for determining λ:

$$\begin{vmatrix} \lambda^2 - A_{11}, & -A_{13}, & -2\omega\lambda - B_{11}, & -B_{13} \\ -A_{31}, & \lambda^2 - A_{33}, & -B_{31}, & -2\omega\lambda - B_{33} \\ 2\omega\lambda - C_{11}, & -C_{13}, & \lambda^2 - D_{11}, & -D_{13} \\ -C_{31}, & 2\omega\lambda - C_{33}, & -D_{31}, & \lambda^2 - D_{33} \end{vmatrix} = 0, \tag{4}$$

$$\begin{vmatrix} \lambda^2 - E_{11}, & -E_{13} \\ -E_{31}, & \lambda^2 - E_{33} \end{vmatrix} = 0. \tag{5}$$

Equation (4) is of the eighth degree in λ, while (5) is a quadratic in λ^2. Liouville [4] found one real root for equation (4). This gave rise to terms of the type $x_i=K_ie^{ht}$, $y_i=L_ie^{ht}$, where h is real and positive; and hence each of the coördinates increases without limit as t increases. He accordingly remarked that if the creator had placed the Moon at one of these straight line positions with reference to the Earth and Sun it would not have stayed there on account of the perturbations of the other planets.

If Liouville had been able to find all the values of λ he would have found $\lambda=0, 0, \pm i\omega, \pm i\rho_1, \pm\rho_2$, from equation (4) and $\lambda=\pm i\omega, \pm i\nu$, from equation (5), where ρ_1, ρ_2, and ν are real numbers.

8. Stability of the equilateral triangle positions. The characteristic equation for the equilateral triangle positions can be set up in a similar manner. Again the z_i equations are independent of x_i and y_i equations so that the values of λ must be obtained from solving an eighth degree equation and a fourth degree equation. All these values of λ have been found [6]. If a certain condition on the masses is satisfied, they are $\lambda=0, 0, \pm i\omega, \pm i\nu, \pm i\rho$, from the x_i and y_i equations, while $\lambda=\pm i\omega$ from the z_1 equation and $\lambda=\pm i\omega$ from the z_3 equation. Hence the only non-periodic terms in the complete solution are those coming from the double zero exponent. Hence the oscillations are of the type

$$x_i = A_{01} + A_{11}t + \text{periodic terms},$$
$$y_i = B_{01} + B_{11}t + \text{periodic terms},$$
$$z_i = \text{periodic terms}.$$

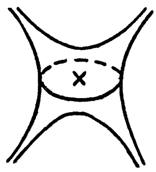
If we exclude the possibility of incommensurability of the periods of the periodic terms then two conditions are enough to insure periodicity, if these two conditions are chosen to make A_{11} and B_{11} vanish.

It may be well here to compare these results with the so-called restricted problem of three bodies in which one of the masses is infinitesimal. In that case, if a certain condition on the masses is satisfied, all the characteristic exponents are pure imaginaries and apparently the equilateral triangle positions are stable. Such a conclusion is not warranted, however, for there is no truly infinitesimal body.

9. The Trojan planetoids. Returning to the problem of the finite masses, we observe that the condition on the masses which must be satisfied in the equilateral triangle case in order that all the characteristic exponents be zero or purely imaginary, is

$$\omega^4 - 27(m_1m_2 + m_1m_3 + m_2m_3) \geqq 0.$$

If m_1 is the mass of the Sun, m_3 is the mass of Jupiter and m_2 is the mass of one of the Trojan planetoids, the above condition is satisfied, but their position is not stable on account of the terms $A_{0i}+A_{1i}t$ which arise from the double zero values of λ. There are 10 such planetoids now known and for some years they have been near the Trojan equilateral triangle points, five in the forward position and five trailing sixty degrees behind the line of Jupiter and the Sun. One cannot help but wonder whether, since none of the planetoids are exactly at the equilateral triangle point, and since there are constant perturbations by the other planets, they will not in time depart from those positions. There are several conjectures as to why they remain there. One possible explanation is that there must be one of the largest accumulations of loose material in our solar system near these points and these planetoids are embedded in this so that the actions of the larger planets are nullified. Another possible explanation is that they are inside a surface of zero relative velocity and cannot escape. One remembers that



it was in this way that Hill showed that our Moon cannot escape from the Earth. Some of the surfaces of zero velocity near these triangle points are shaped roughly like the adjacent figure. If a planetoid were inside such a surface it could not escape but could vary quite considerably in its distance from the triangle point.

10. Periodic orbits. The question arises as to whether or not there may be special initial conditions for which there are periodic orbits near the straight line and equilateral triangle positions. This problem was solved by F. R. Moulton [7] and some of his students for an infinitesimal body. His results have been extended to the case where all the bodies are finite in papers by H. E. Buchanan which appeared in the *American Journal* in 1923, 1925, 1927, and 1928 and to which reference has already been made, one type, in each case, being the ellipses which Lagrange found.

So far we have mentioned only particular solutions of equations (1). These equations have been solved in general by Sundman [8], who started from some results of Levi-Civita and gave a complete formal solution in series convergent for all real values of t. F. R. Moulton also outlines a method for obtaining a complete formal solution in his book on *Differential Equations* [9]. Their work is of the highest importance from a mathematical standpoint, but its practical value from the point of view of the astronomer is not very great. Their solutions give no properties of the motion, no shapes of the orbits, no proof of periodicity or of non-periodicity and they have very little prospect of ever being applied.

11. The helium atom. Now we turn to another type of problem. In 1927, U. Crudeli [10] published a twenty-page article on the stability of the equilateral triangle positions in the so-called neutral helium atom. The helium atom is made up of a central mass with a positive charge of $2e$ and two equal small masses with charges of $-e$ each. The equilateral triangle positions are quite different from those of the ordinary three body problem. The triangle rotates about its center of gravity but the nucleus and electrons move in circles whose planes are perpendicular to the plane of the triangle. Crudeli investigated the stability of these positions and found one real root of the characteristic equation, which was of eighteenth degree. His conclusion was that the positions were unstable. In 1931 and 1933 in two papers in this MONTHLY [11] I discussed the stability of the triangle and straight line positions for the helium atom after showing that such particular solutions exist for finite values of the masses. In both of these papers only the forces due to the electrical charges were considered. All the characteristic exponents were found; they are of the form:

(a) in the triangle case, $\lambda=0,\ 0,\ \pm i\omega,\ \pm(\alpha\pm i\beta),\ \pm i\nu,\ \pm i\rho$;

(b) in the straight line case, $\lambda=0,\ 0,\ \pm i\omega,\ \pm i\nu,\ \pm\rho,\ \pm i\sigma,\ \pm i\omega$,

the last two coming from the z_i equations which are independent of the x_i, y_i equations. Hence both positions are unstable.

12. The characteristic exponents. It is rather remarkable that the exponents $0,\ 0,\ \pm i\omega$ appear again in both cases. Their appearance in four different cases leads one to suspect a common underlying reason. Poincaré showed, about forty years ago, that the existence of an integral of equations (2), independent of t and satisfying certain conditions on its partial derivatives, required that one characteristic exponent be zero. There are two such integrals in the three body problem and also two in the helium atom, so in all four cases one could predict that the zeros must occur. This fact is not of much help in solving the characteristic equation, for the lowest degree in λ^2 in any of the four problems is four. The knowledge that there is a zero root in λ^2 would reduce the degree to the third and this is practically impossible to solve with the complicated coefficients which occur.

In a paper which appeared in the *Duke Journal* [12] in 1935 it was proved that if the equations of variation have an integral whose minimum period is $2\pi/\omega$ then one characteristic exponent must be either $+i\omega$ or $-i\omega$. There are two such integrals in each case, hence the characteristic exponents $\pm i\omega$ must appear. This knowledge is sufficient to reduce the degree of the characteristic equation to the second in most of the cases.

13. The generalized helium atom. The characteristic equations in the four cases mentioned above had already been solved before the results of the paper in the *Duke Journal* of December 1935 were known. However, two [13] interesting applications have occurred since then. If, in the helium atom, we consider the gravitational forces as well as the forces due to the electrical charge, then there are straight line solutions and isosceles triangle solutions. The equations of variation from these positions have four integrals of the same type as before. Consequently the characteristic equations must have the roots 0, 0, and $\pm i\omega$. Without this knowledge it would, I think, have been impossible to solve the characteristic equation for these cases since the difficulty of the algebra is at least double that for the four simpler cases. With these new results at hand the characteristic exponents for these more complicated cases have been found and the stability of the straight line and isosceles triangle positions for the generalized helium atom have been completely discussed. One of the papers in which this was done has not yet been published.*

14. Conclusion. A mathematician never needs to apologize for studying any problem, however useless. The helium atom we have discussed has long ago passed from the mind of the physicist as a possible explanation of the interior mechanism of any atom. But the results of the above papers have recently been used in studying the stability of certain positions in the problem of four bodies. The positions in the four body problem to which I refer were established by MacMillan and Bartky [14] in 1932. One position in the four body problem is that in which the bodies are at the vertices of a square. When the equations of variation from this position are set up, as indicated above, their characteristic equation must have the four roots 0, 0, and $\pm i\omega$.

Many valuable contributions have not been mentioned because they did not seem to be a part of the particular tale I have been telling. The problem of three bodies is not yet solved as thoroughly and completely as we wish but after all it is only about two hundred and fifty years old!

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KASNER'S INVARIANT AND TRIHORNOMETRY

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This paper contains a new proof of a theorem of Kasner on plane conformal geometry. The theorem, it is shown at the end, holds not only in the plane but on any curved surface. It follows that the theorems of "trihornometry" apply to horn angles on any surface.

1. Introduction. Conformal geometry studies those properties of geometrical objects which cannot be changed by any conformal transformation. A conformal transformation can be defined as an analytic homeomorphism between a region

* Miss Adrienne Rayl solved this problem while a student in the author's class in Celestial Mechanics.